SOME RESULTS ON MEASURABILITY IN GAMBLING PROBLEMS

by

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1. Introduction.

Let Γ be a measurable gambling house defined on a Borel set of fortunes F. Starting with fortune f, a gambler chooses a strategy σ available to him. The strategy σ induces a probability measure on the product space $H = F \times F \times \cdots$ of histories of fortunes and the gambler is paid $\int g d\sigma$, the expectation of g under σ, where g is some utility function on H. Let $M_g(f)$ be the sup $\int g d\sigma$ taken over all measurable strategies "essentially" available at f and let $\Gamma_g(f)$ be the same supremum taken over all strategies σ available at f. The function M_g is well-defined when g is a bounded, Borel measurable function and it is shown below that, in this case, M_g is universally measurable. The function Γ_g is well-defined if g is bounded and finitary. If g is bounded, finitary, and Borel, then both functions are well-defined and seen to be equal. Thus a gambler can do just as well when restricted to measurable strategies for these problems.

These results seem to contain most of the known results on the measurability of the return function and the adequacy of measurable strategies (the techniques used owe much to those originated by Strauch), but the problem which motivated this research nevertheless remains open. That is, do good measurable strategies exist for measurable problems with a measurable utility function of the type studied by Dubins and Savage? (If so, the return function is measurable.)

Some progress is made on this question. Let u be a bounded function on F and σ a strategy. Then $u(\sigma)$ is defined to be $\limsup_{t \to \infty} \int u(f_t) d\sigma$, where the lim sup is over all stop rules t. It is shown that for u and σ measurable, it is equivalent to take the lim sup over all measurable stop rules and also $u(\sigma) = \int u^* d\sigma$ where u^* is a bounded, measurable function on H. By the result previously mentioned, Mu^* is universally measurable. The question remaining is whether Mu^* is the optimal return function V studied by Dubins and Savage.

For expository reasons, the results for $u(\sigma)$ are presented first. However, the reader who wishes may skim section 2 and skip to sections 4 and 5 for the results outlined in the first paragraph.

2. Measurable Strategies.

Let F be a set and let G be the set of all gambles on F. That is, G is the set of all finitely additive probability measures defined on all subsets of F. A strategy σ is a sequence $\sigma_0, \sigma_1, \cdots$ where $\sigma_0 \in G$ and, for $n \geq 1$, σ_n maps $F \times \cdots \times F$ (n-factors) into G. Let H be the countably infinite product $F \times F \times \cdots$ and let g be a bounded, finitary function on H. Then $\int g d\sigma$ was defined in [2].

Now suppose $\mathcal{B}$ is a Borel field of subsets of F. Let σ be a strategy and suppose σ_0 restricted to $\mathcal{B}$ is countably additive and, for every $n \geq 1$ and every n-tuple $(f_1, \cdots, f_n)$ of elements of F, $\sigma_n(f_1, \cdots, f_n)$ is countably additive. Suppose also that, for every $n \geq 1$ and every $B \in \mathcal{B}$, $\sigma_n(f_1, \cdots, f_n)(B)$ is a $\mathcal{B} \times \cdots \times \mathcal{B}$ (n-factors) - measurable function of $(f_1, \cdots, f_n)$. Then σ is said to be a measurable strategy.

Denote by $\mathcal{B}^\infty$ the product Borel field $\mathcal{B} \times \mathcal{B} \times \cdots$ of subsets of H. A measurable strategy σ naturally induces a countably additive measure $\mu = \mu(\sigma)$ on $\mathcal{B}^\infty$. That is, the μ-marginal distribution of

f_1 is σ_0 and, for every $(f_1, \cdots, f_n)$, the conditional μ-distribution of f_{n+1} given $(f_1, \cdots, f_n)$ is $\sigma_n(f_1, \cdots, f_n)$. (Here σ_0 and $\sigma_n(f_1, \cdots, f_n)$ have been tacitly identified with their restrictions to $\mathcal{B}$. Where it is not harmful, such identifications will be made below also.)

The relationship between the measure σ defined on the finitary subsets of H and the measure μ defined on $\mathcal{B}^{\infty}$ was discussed in [10], but a simpler and more general analysis is given here. Some related work is also in [5].

__Theorem 1__: _Let g_1 be a bounded, finitary function on H and let g_2 be a bounded, $\mathcal{B}^{\infty}$-measurable function on H. If σ is a measurable strategy and $g_1 \leq g_2$, then $\int g_1 d\sigma \leq \int g_2 d\mu$. (Hence, $g_1 \geq g_2$ implies $\int g_1 d\sigma \geq \int g_2 d\mu$.)_

__Proof__: The proof is by induction on the structure of g_1. The theorem is certainly true for g_1 of structure 0; i.e. for g_1 a constant function. Now assume it is true for functions of structure less than α and suppose the structure of g_1 is α. If $g_1 \leq g_2$, then $g_1 f_1 \leq g_2 f_1$ for all f_1 in F (recall that, for any function g on H and f in F, gf is the function on H defined by $gf(f_1, \cdots) = g(f, f_1, \cdots)$). Let $\mu[f_1]$ denote the measure on $\mathcal{B}^{\infty}$ induced by the conditional strategy $\sigma[f_1]$. Then $\mu[f_1]$ is a version of the regular conditional distribution of μ given f_1 and, using the inductive assumption, we have

$$\int g_1 d\sigma = \int \{\int (g_1 f_1) d\sigma[f_1]\} \, d\sigma_0(f_1) \leq \int \{\int (g_2 f_1) d\mu[f_1]\} \, d\sigma_0(f_1)$$

$$= \int g_2 d\mu. \ \square$$

__Corollary__: _If g is bounded, finitary, and measurable, then $\int g d\sigma = \int g d\mu$._

The following proposition is elementary.

Proposition: *Let* L *be a linear space of functions on a set* H *and let* X *and* Y *be linear subspaces. Suppose* S *and* T *are linear functionals on* X *and* Y, *respectively, such that, for every* $x \in X$ *and* $y \in Y$, *if* $x \leq y$, *then* $Sx \leq Ty$. *Then there is a non-negative linear functional* V *on* L *such* V *restricted to* X *is* S *and* V *restricted to* Y *is* T.

Proof: For $x \in X$ and $y \in Y$, let $V(x + y) = Sx + Ty$. Then V is easily seen to be well-defined on the space spanned by X and Y and non-negative there. Now extend V to the rest of L. $\square$

It follows from Theorem 1 and the proposition that there is a finitely additive probability measure which extends both σ and μ. Abusing language for the sake of convenience, we shall denote by σ one such extension. Thus σ restricted to $\mathcal{B}^\infty$ is μ and σ is countably additive on $\mathcal{B}^\infty$.

Now if σ is a measurable strategy and $p = (f_1, \cdots, f_n)$ is any partial history, then the conditional strategy $\sigma[p]$ is measurable. Thus $\sigma[p]$ induces a countably additive measure on $\mathcal{B}^\infty$ which, using the agreed upon convention, we again denote by $\sigma[p]$. Let t be any stop rule and recall that $p_t(h) = (f_1, \cdots, f_{t(h)})$, where, as usual, $h = (f_1, f_2, \cdots)$. The formula

$$(1) \qquad \int g d\sigma = \iint (g p_t(h)) d\sigma[p_t(h)] d\sigma(h)$$

was given in [2], p.51, for g bounded and finitary. If σ is measurable, then (1) also holds for every g which is bounded and measurable. The proof, as suggested in [2], is by induction on the structure of p_t. The formula specializes to give

(2) $\qquad \sigma(A) = \int \sigma[p_t(h)](Ap_t(h))\, d\sigma(h)$

where $A(f_1, \cdots, f_n) = \{(f_{n+1}, \cdots):(f_1, \cdots, f_n, f_{n+1}, \cdots) \in A\}$. This equation holds for A finitary or measurable. Since every countably additive probability measure on $\mathcal{B}^\infty$ is induced by some measurable strategy, these formulae may have some interest other than their application in the next section.

3. The Dubins and Savage utility of a measurable strategy.

In this section, σ is a fixed measurable strategy and u is a bounded $\mathcal{B}$-measurable function on F. The function u is called the utility function and Dubins and Savage [2] define the utility of σ by $u(\sigma) = \limsup_{t \to \infty} u(\sigma, t)$, where $u(\sigma, t) = \int u(f_t)\, d\sigma$ and the lim sup is taken over all stop rules t. Denote by $\bar{u}(\sigma)$ the same lim sup taken over $\mathcal{B}^\infty$-measurable stop rules t. (Stop rules are taken to be everywhere finite in [2], but $\bar{u}(\sigma)$ would be the same if we permitted stop rules which are finite almost surely.) The result of this section is a formula for $u(\sigma)$ and $\bar{u}(\sigma)$, which also proves that they are equal. (The interested reader can see that our argument proves that $\limsup_{t \to \infty} u(\sigma, t)$ taken over all measurable, positive integer-valued functions has the same value.)

Most of the work is done in the case when u is non-negative and simple. So, until further notice, assume u is of the form

(1) $\qquad \sum_{i=1}^{m} a_i \, 1_{A_i} \qquad \text{where } a_1 > \cdots > a_M > 0$

and the A_i are pairwise disjoint sets in $\mathcal{B}$.

Theorem 1: *Let u be as in (1) and let $B_i = \{(f_1, f_2, \cdots):f_k \in A_i$ for infinitely many $k\}$ for $i = 1, \cdots, M$. Then*

$$u(\sigma) = \bar{u}(\sigma) = a_1\sigma(B_1) + a_2\sigma(B_2 - B_1) + \cdots + a_m\sigma(B_M - (B_1 \cup \cdots \cup B_{M-1})).$$

This theorem is a generalization of Theorem 2 of [10]. The proof will be given in several lemmas.

Let $h = (f_1, f_2, \ldots)$ and for each positive integer N and for $i = 1, \cdots, M$, define

$$E_i^N = \{h: \; f_k \in A_i \;\text{ for some }\; k \geq N\},$$

$$F_i^N = \bigcup_{j=1}^{i} E_j^N, \quad F_0^N = \emptyset.$$

Suppose the gambler must play for N days and then may stop whenever he pleases. He would prefer to stop at a fortune in A_1 and receive a_1; the next best thing would be to stop in A_2 and receive a_2; and so forth. This idea suggests our first lemma.

__Lemma 1__: __For every positive integer__ N,

$$\sup_{t \geq N} u(\sigma, t) \leq \sum_{i=1}^{M} a_i \sigma(E_i^N - F_{i-1}^N).$$

__Here, the supremum may be taken over all integer - valued functions__ t __for which__ $u(f_t)$ __is__ σ-__integrable__.

__Proof__: If $t \geq N$, then

$$\begin{aligned} u(\sigma, t) &= \int u(f_t)\, d\sigma \\ &= \int_{E_1^N \cup \cdots \cup E_M^N} u(f_t)\, d\sigma \\ &= \sum_{i=1}^{M} \int_{E_i^N - F_{i-1}^N} u(f_t)\, d\sigma \\ &\leq \sum_{i=1}^{M} a_i \sigma(E_i^N - F_{i-1}^N). \quad \square \end{aligned}$$

Now let B_i be as in Theorem 1 and define, for $i = 1, \ldots, M$,

$$C_i = \bigcup_{k=1}^{i} B_k, \quad C_0 = \emptyset,$$

$$D_i = B_i - C_{i-1},$$

$$E = \sum_{i=1}^{M} a_i \sigma(D_i).$$

If the gambler must play for an arbitrarily long time, it would be best to have a history h in B_1. For then he can find fortunes in A_1 arbitrarily far in the future. If B_1 is not possible, he would prefer B_2 and so on as reflected in the next lemma.

Lemma 2: $u(\sigma) \leq E$ and $\overline{u}(\sigma) \leq E$.

Proof: Notice that, for every positive integer N, $u(\sigma) \leq \sup_{t \geq N} u(\sigma, t)$. Also, the sets E_i^N and F_{i-1}^N decrease to B_i and C_{i-1}, respectively, as $N \to \infty$. These sets are $\mathcal{B}^\infty$ measurable and σ is countably additive on $\mathcal{B}^\infty$. Hence, $\sigma(E_i^N - F_{i-1}^N) \to \sigma(B_i - C_{i-1})$ as $N \to \infty$ for $i = 1, \cdots, N$. Now apply Lemma 1 to see $u(\sigma) \leq E$.

The proof for $\overline{u}(\sigma)$ is the same. $\square$

The next two lemmas help to establish the reverse of the inequalities in Lemma 2.

Lemma 3: For every positive integer N, $\sup_{t \geq N} u(\sigma, t) \geq E$. Here the supremum may be taken over all measurable stop rules t.

Proof: We construct a measurable stop rule t such that $t \geq N$ and $u(\sigma, t)$ is arbitarily close to E. Let $\varepsilon > 0$.

Set $N_0 = N$ and inductively define measurable, (possibly) incomplete stop rules $t_1, \cdots, t_M$ and integers $N_1, \cdots, N_M$ as follows: For $i = 1, \cdots, M$, let

$$t_i(h) = \text{the first } k \geq N_{i-1} \text{ such that } f_k \in A_i$$
$$= \infty \text{ if there is no such } k.$$

Using countable additivity, choose $N_i > N_{i-1}$ so that $\sigma[t_i < \infty] \leq \sigma[t_i < N_i] + \mathcal{E}$.

Then, for $i = 1, \cdots, M$, $[t_i < \infty] \supseteq B_i \supseteq D_i$.

Hence,

$$\sigma(D_i \cap [t_i \geq N_i]) \leq \sigma([t_i < \infty] \cap [t_i \geq N_i]) \leq \mathcal{E},$$

and

$$\sigma(D_i \cap [t_i < N_i]) \geq \sigma(D_i) - \mathcal{E}.$$

Now define

$$t = t_1 \wedge \cdots \wedge t_M \wedge N_M.$$

Then t is a measurable stop rule and $t \geq N$. Notice that if $t_i < N_i$, then $t < N_i$ and $t = t_j$ for some $j \leq i$. So $u(f_t) = u(f_{t_j}) = a_j \geq a_i$.

Now we can compute

$$\begin{aligned} u(\sigma, t) &\geq \sum_{i=1}^{M} \int_{D_i} u(f_t)\, d\sigma \\ &\geq \sum_{i=1}^{M} \int_{D_i \cap [t_i < N_i]} u(f_t)\, d\sigma \\ &\geq \sum_{i=1}^{M} a_i \{\sigma(D_i) - \mathcal{E}\}. \end{aligned}$$

Since $\mathcal{E}$ was arbitrary, the proof is complete. □

<u>Lemma 4</u>: <u>For any stop rule</u> s, $\sup_{t \geq s} u(\sigma, t) \geq E$. <u>Here the supremum is over all stop rules</u> t.

Proof: Let $\mathcal{E} > 0$.

Let $p = (f_1, \cdots, f_n)$ be a partial history. By Lemma 3, there is, for every p, a stop rule $t(p)$ such that

$$u(\sigma[p], t(p)) \geq \sum_{i=1}^{M} a_i \sigma[p](D_i) - \varepsilon$$

$$= \sum_{i=1}^{M} a_i \sigma[p](D_i p) - \varepsilon .$$

The last equation uses the fact that $D_i p = \{(f_{n+1}, \cdots):(f_1, \cdots, f_n, f_{n+1}, \cdots) \in D_i\} = D_i$, for every p and for $i = 1, \cdots, M$.

Recall that $p_s(h) = (f_1, \cdots, f_{s(h)})$ and let t be the stop rule which is the composition of s with the family $t(p_s(h))$. That is,

$$t(h) = s(h) + t(p_s(h))(f_{s(h)+1}, \cdots)$$

for h in H. Then, by formulae (1) and (2) of section 2,

$$\begin{aligned} u(\sigma, t) &= \int u(\sigma[p_s(h)], t(p_s(h)))\, d\sigma(h) \\ &\geq \int \{ \sum_{i=1}^{M} a_i \sigma[p_s(h)](D_i p_s(h)) - \varepsilon \}\, d\sigma(h) \\ &= \sum_{i=1}^{M} a_i \sigma(D_i) - \varepsilon . \ \square \end{aligned}$$

Since $u(\sigma) = \inf_s \sup_{t \geq s} u(\sigma, t)$, it follows from Lemma 4 that $u(\sigma) \geq E$. Now $\overline{u}(\sigma)$ equals the same expression except that the infimum and supremum are over measurable stop rules. Let s be a measurable stop rule. Then, for any $\varepsilon > 0$, there is an integer N such that $\sigma[s \leq N] > 1 - \varepsilon$. It is easy to see that $\sup_{t \geq s} u(\sigma, t) \geq \sup_{t \geq N} u(\sigma, t) - 2\varepsilon \sup|u|$, and, hence,

$$\overline{u}(\sigma) = \inf_N \sup_{t \geq N} u(\sigma, t).$$

So, by Lemma 3, $\overline{u}(\sigma) \geq E$.

This completes the proof of Theorem 1.

Now to any real-valued function u on F, we associate a function u^* on H defined by

$$u^*(f_1, f_2, \cdots) = \limsup_{n\to\infty} u(f_n).$$

Theorem 2: Let u be a bounded, measurable function on F. Then $u(\sigma) = \overline{u}(\sigma) = \int u^* d\sigma$.

Proof: There is no loss of generality in assuming u is non-negative. (If not, add a sufficiently large constant to u and check that no harm is done.) So assume $u \geq 0$ and choose a sequence u_n of non-negative simple functions converging uniformly to u. The desired equations hold for the u_n by Theorem 1. Now pass to the limit. $\square$

After I showed Theorem 1 to Lester Dubins, he pointed out that the expression there could be interpreted as the integral of u^* as in Theorem 2.

A digression on a "Fatou equation"

The previous result shows that in the setting above, the integral and lim sup over the directed set of stop rules commute. The usual inequality of Fatou might lead us not to expect equality in general. Here we show that the setting above is fairly general and that Theorem 2 includes the following result.

Theorem 3: Let $X_1, X_2, \cdots$ be a uniformly bounded sequence of random variables on a probability space and let $Y^* = \limsup_{n \to \infty} X_n$. Then $EY^* = \limsup_{t \to \infty} EX_t$ where the lim sup is taken over the set of measurable stop rules with respect to the X_n-process.

Proof: Let B be a uniform bound on the X_n. We can assume that the probability space is the space of sequences $\Omega = \{(x_1, x_2, \cdots): |x_i| \leq B, i = 1, 2, \cdots\}$, and X_n is the n^{th} coordinate map.

Set $f_n = (x_1, \cdots, x_n)$. Let σ_0 be the distribution of X_1 and let $\sigma_n(f_1, \cdots, f_n)$ be the conditional distribution of $(X_1, \cdots, X_{n+1})$ given $(X_1, \cdots, X_n) = (x_1, \cdots, x_n)$. Thus we are, in effect, taking F to be all finite sequences $(x_1, \ldots, x_n)$ with $|x_i| \le B$ for all i. Define the utility function by $u(x_1, \cdots, x_n) = x_n$.

There is a natural correspondence between stop rules. Let s be a stop rule on H. The corresponding stop rule t on Ω is given by

$$t(x_1, x_2, x_3, \cdots) = s(x_1, (x_1, x_2), (x_1, x_2, x_3), \cdots).$$

Since σ gives probability 1 to histories of the form $(x_1, (x_1, x_2), \cdots)$, the map is essentially a one-one correspondence. Moreover, the random variables $u(f_s)$ and X_t have the same distribution. Likewise u^* and Y^* have the same distribution. The desired result now follows from Theorem 2. □

I have not checked whether the boundedness assumption is essential, but one would hope that it could be weakened somewhat.

4. A lemma.

In this section, a lemma needed for the sequel is proved and a few conventions are made about notation.

If $(X, \mathcal{B}(X))$ is a measurable space, denote by $\mathcal{P}(X)$ the collection of countably additive probability measures defined on $\mathcal{B}(X)$ and let $\Sigma(X)$ be the smallest σ-field of subsets of $\mathcal{P}(X)$ which makes $p \to p(A)$ a measurable function of p for each A in $\mathcal{B}(X)$ (cf. [1]).

Lemma: *Let $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ be measurable spaces and let g be a real-valued measurable function defined on the product $X \times Y$. Suppose also that g is bounded from below (say). Then the map*

$$(x, p) \to \int g_x\ (y)\ dp(y)$$

is measurable from $X \times \mathcal{P}(Y)$ *to the reals.* (*Here* $g_x(y) = g(x, y)$.)

Proof: Choose a sequence g_n of simple functions on $X \times Y$ such that $g_n \uparrow g$. Then, for every (x, p),

$$\int (g_n)_x dp \uparrow \int g_x dp.$$

Thus we can assume g is simple and, in fact, we may as well assume $g = 1_A$ is the indicator of a measurable subset of $X \times Y$.

Now let $\mathcal{A}$ be the collection of sets A for which $(x, p) \to p(A_x)$ is measurable. (Here, $A_x = \{y:(x, y) \in A\}$.) Then $\mathcal{A}$ is a monotone class which contains the field of disjoint unions of measurable rectangles. $\square$

(The lemma could be stated and proved with $M(Y)$ (the set of finite signed measures on $\Sigma(Y)$) in place of $\mathcal{P}(Y)$ and would then be a generalization of 2.2 of [1].)

In what follows, we deal mainly with *Borel sets*, which we take to mean a Borel subset of a complete separable metric space. Let X be a Borel set and $\mathcal{B}(X)$ the Borel subsets of X. If $\mathcal{P}(X)$ is given the usual weak topology (see, for example, chapter II of [4]), then $\mathcal{P}(X)$ has the structure of a Borel set. It is not difficult to see that the σ-field of Borel subsets of $\mathcal{P}(X)$ is the σ-field $\Sigma(X)$ defined above.

5. Basic results on measurability.

In the remainder, we assume the setting of a measurable gambling house as defined in [7]. That is, F is assumed to be a Borel subset of a complete, separable metric space and $\mathcal{B}$ to be the Borel subsets of F. A gambling house Γ on F assigns to each f in F a non-void set $\Gamma(f)$ of gambles γ defined on all subsets of F. Let $p(\gamma)$

denote the restriction of any gamble γ to $\mathcal{B}$. The house Γ is called <u>measurable</u> if, for every f in F and every γ in $\Gamma(f)$, $p(\gamma)$ is countably additive and if the set $\Gamma' = \{(f, p(\gamma)):\gamma \in \Gamma(f)\}$ is in the product σ-field $\mathcal{B} \times \Sigma(F)$. In the sequel γ is often written for $p(\gamma)$.

A strategy σ is <u>available</u> at f in Γ if $\sigma_0 \in \Gamma(f)$ and, for every $(f_1,\cdots, f_n)$, $\sigma_n(f_1,\cdots, f_n) \in \Gamma(f_n)$. A natural class of stochastic processes is the set of measurable strategies available at some f in Γ. This set may be empty, however (see [8] and [9]). So we consider the measurable strategies σ such that $\sigma_0 \in \Gamma(f)$ and, for every $n > 0$, $\sigma_n(f_1,\cdots, f_n) \in \Gamma(f_n)$ σ-almost surely. Such a σ is said to be <u>essentially available</u> at f in Γ.

<u>Remark</u>: As pointed out in [9], there are measurable strategies available in Γ iff there is a measurable map $\alpha:F \to \mathcal{P}(F)$ such that $\alpha(f) \in \Gamma(f)$ for every f. If such a map exists, then every measurable strategy σ which is essentially available at f in Γ induces the same distribution on H as some measurable strategy $\bar{\sigma}$ available at f in Γ. To get $\bar{\sigma}$, just change the values of the $\sigma_n(f_1,\cdots, f_n)$ to be $\alpha(f_n)$ whenever $\sigma_n(f_1,\cdots, f_n) \notin \Gamma(f_n)$.

Denote by $\Gamma^\infty(f)$ the set of all measurable strategies σ essentially available at f in Γ. Let $\mu(\sigma)$ be the probability measure induced by σ on $\mathcal{B}^\infty$ (see section 2) and set

$$\Gamma^\infty = \{(f, \mu(\sigma)):\sigma \in \Gamma^\infty(f)\}.$$

Recall that $\mathcal{P}(H)$ denotes the set of probability measures on the Borel sets $\mathcal{B}^\infty$ of H.

Theorem 1: The set Γ^{∞} is a Borel subset of $F \times \mathcal{P}(H)$.

Proof: Essentially the same as Theorem 2.1 of [9]. $\square$

Now let g map H to the reals. Think of g as a utility function. For g bounded below (say) and measurable, we can study the optimal return function

$$M_g(f) = \sup \int g d\sigma,$$

where the supremum is over all $\sigma \in \Gamma^{\infty}(f)$.

Theorem 2: The function M_g is universally measurable (i.e. measurable with respect to the completion of any measure on the Borel sets of F) if g is bounded below and measurable.

This result is closely related to Strauch's Theorem 7.1 in [6] which our theorem includes. Using Theorem 1, we could essentially repeat Strauch's proof. Instead we prove Theorem 3 below which includes Theorem 2.

Examples: To illustrate Theorem 2 above, let r_n be a sequence of non-negative measurable functions on F. In gambling and dynamic programming problems, the function g is often one of the following forms:

$$\Sigma\, r_n(f_n),\ \limsup r_n(f_n),$$

$$\text{or} \quad \limsup \frac{r_1(f_1)+\dots+r_n(f_n)}{n}.$$

Now recall that $(gf)(f_1,\cdots) = g(f, f_1,\cdots)$ by definition.

Theorem 3: The function $f \to M_{(gf)}(f)$ is universally measurable if g is bounded below and measurable.

Proof: For any real number a, the set $\{f : M_{(gf)}(f) > a\}$ is the projection of the set $\{(f, p) : p \in \Gamma^{\infty}(f), \int (gf) dp > a\}$. By Theorem 1 and the lemma of section 4, the latter set is Borel. So its projection is analytic and, hence, universally measurable by a famous result of Kuratowski. $\square$

Now, for any bounded, finitary function g on H and f in F, set

$$\Gamma_g(f) = \sup \int g d\sigma,$$

where the supremum is over all strategies σ available at f.

__Theorem 4__: __If__ g __is bounded__, __finitary__, __and measurable__, __then__ $\Gamma_g = M_g$.

__Proof__: Certainly, $M_g \leq \Gamma_g$ since any strategy σ essentially available induces the same distribution on H as some strategy σ', which is available. To get σ', just change σ to make it available on those partial histories where it is not. By assumption, the changes take place on a set of σ-measure zero.

The proof of the opposite inequality is by induction on the structure of g. The result is clear for g of structure zero. So assume it is true for all functions of structure less than α and suppose g has structure α.

Fix f in F and $\varepsilon > 0$. Choose a strategy σ available at f such that

$$\int g d\sigma > \Gamma_g(f) - \varepsilon .$$

Then notice that

$$\begin{aligned}\int g d\sigma &= \int \{\int (gf_1) d\sigma[f_1]\} \, d\sigma_0(f_1) \\ &\leq \int \Gamma_{(gf_1)}(f_1) \, d\sigma_0(f_1) \\ &= \int M_{(gf_1)}(f_1) \, d\sigma_0(f_1).\end{aligned}$$

The last equation is by the inductive assumption. Now, by Theorem 6.3 of [3], there is a measurable map $\bar{\sigma}: F \to \mathcal{P}(H)$ such that

$$\sigma_0\{f_1 : \bar{\sigma}(f_1) \in \Gamma^{\infty}(f_1) \text{ and } \int (gf_1) d\bar{\sigma}(f_1) > M_{(gf_1)}(f_1) - \mathcal{E}\} = 1.$$

(To apply the theorem cited, use the fact that, by Theorem 3, $M_{(gf_1)}(f_1)$ differs from some Borel measurable function on a set of σ_0 measure zero.) Now let σ' be that measurable strategy whose initial gamble is σ_0 and whose conditional strategy given f_1 corresponds to $\bar{\sigma}(f_1)$. Then σ' is essentially available at f, and we have

$$M_g(f) \geq \int g d\sigma' = \int \{\int (gf_1) d\bar{\sigma}(f_1)\} \, d\sigma_0(f_1)$$

$$\geq \int \{M_{(gf_1)}(f_1) - \mathcal{E}\} \, d\sigma_0(f_1) \geq \Gamma_g(f) - 2\mathcal{E}. \; \Box$$

Theorem 4 generalizes Theorem 2.2 of [9], which, as pointed out there, implies Strauch's result in [7] on the measurability of the return function V for measurable, leavable gambling problems.

Now consider a measurable gambling problem with a utility function of the type studied in section 3. That is, let u be a bounded, measurable function on F and let $u(\sigma)$ be as in section 3. For f in F, let $Q(f)$ be the sup $u(\sigma)$ taken over all measurable strategies σ essentially available at f in Γ, and let $V(f)$ denote the same supremum taken over all strategies σ available at f in Γ.

<u>Theorem 5</u>: <u>The function Q is universally measurable</u>.

<u>Proof</u>: By Theorem 2 of section 3, we see that $Q = Mu^*$. Now apply Theorem 2 of this section. $\Box$

The question remaining is whether $Q = V$. Certainly, $Q \leq V$ (see the first paragraph of the proof of Theorem 4). Thus to prove equality, it is enough to find nearly optimal measurable strategies essentially available. Equality was proved for leavable houses by Strauch in [7] and for houses with a goal by the author in [10]. The construction of good measurable strategies given in [10] can, with some effort, be adapted to handle the case when u is the indicator of a countable set. It is possible that a more sophisticated adaptation would handle the general case.

One is also tempted to apply Theorem 4 since $Q = Mu^*$. Unfortunately, u^* is not finitary and Γu^* has not been given a meaning.

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